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New Approach to Attitude Determination Using Global Positioning System Carrier Phase Measurements

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A new approach to the problem of attitude determination using the global positioning system, which parameterizes the rows of the attitude matrix, differing itself from other methods (i.e., the method of parallel to Wahba's problem (PWP) or the traditional linearized method), is presented. The new approach is a forward procedure and does not need any initial attitude knowledge. Thus, it is more computationally economical than the linearized method, which employs an iterative procedure. Unlike either the linearized method or the PWP method, which require at least two baselines, the approach here is suitable for any baseline configurations, including not only either coplanar or noncoplanar baseline configurations, but also one-baseline configurations. If the antennas are carefully arranged to satisfy the symmetric condition, the derived solution can be further simplified. The concept of the symmetric condition is applicable for any baseline configurations, whereas the concept of the balanced condition proposed in the PWP method is limited to the case of noncoplanar baseline configurations. Both concepts are identical in the noncoplanar case. Therefore, the symmetric condition can be regarded as a natural generalization of the balanced condition. The results of experiments demonstrate that algorithms derived from this new approach are highly efficient.

Introduction

DURING the past 10 years, many authors have investigated the problem of attitude determination using the global positioning system (GPS). Because of its advantages of long-term stable accuracy, low cost, and low power consumption, GPS has the potential to be the key system for spacecraft and aircraft attitude determination and navigation.[1,2] The algorithms of attitude determination using GPS can be categorized into three types[3–6]: those that employ the carrier phase measurements directly, those utilizing the GPS vectorized observations, and those based on a stochastic filtering method. The two former types belong to the point estimation methods. The use of GPS vectorized observations can be summarized as a problem of two-level optimal estimation, and certain conditions must be satisfied to guarantee that the solution is globally optimal.

Two traditional approaches to the first type of algorithm are the nonlinear least-squares fit (NLLSFit) method and the parallel to Wahba's problem (PWP). NLLSFit parameterizes the vector of small-angle rotations and employs an iterative procedure to obtain the attitude solution.[2] When the problem is converted to Wahba's problem, the PWP method can share many algorithms for solving Wahba's problem.[4,7] However, it usually requires at least three noncoplanar baselines to satisfy the balanced condition.[7] Some modified PWP algorithms have been proposed, that is, the method for two baselines.[5]

This paper presents a new approach, which is characterized by its calculating the attitude matrix element solution (AMES) directly. The key idea is to convert a nonlinear problem of attitude angles into a linear weighted least-squares problem of attitude matrix elements. The advantages of this method are that it is less computationally intensive, that it does not need any initial attitude knowledge due to a noniterative procedure, and that it is suitable for both coplanar and noncoplanar baseline configurations as well as for a two-antenna configuration. The derived solution can be further simplified if the antennas are carefully arranged to satisfy the symmetric condition. As is well known, the balanced condition proposed in the PWP method is limited to the case of noncoplanar baseline configurations; however, the concept of the symmetric condition is suitable for any baseline configurations. Furthermore, both concepts become identical in the noncoplanar case. Therefore, the symmetric condition can be regarded as a natural generalization of the balanced condition. Results of experiments using a Trimble Advanced Navigation System (TANS) Vector GPS receiver show that algorithms derived from the new approach are highly efficient.

Statement of Problem

The basic measurement for GPS-based attitude determination is the single difference carrier phase, which is the carrier phase difference between the GPS signals received by two antennae separated by a baseline. This kind of measurement also reflects the projection of the baseline vector onto the line-of-sight (LOS) vector of a GPS satellite.

If there are m baselines and n visible satellites, one method of attitude determination is to find the attitude matrix $\boldsymbol{A}$ that minimizes the following cost function:

$$J(\boldsymbol{A}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2\left(\Delta\phi_{ij} + n_{ij}\cdot\lambda - \beta_j - \hat{\boldsymbol{s}}_i^T\boldsymbol{A}^T\boldsymbol{b}_j\right)^2 \qquad (1)$$

where $\Delta\phi_{ij}$ is the single difference carrier phase measurement in length units of wavelengths corresponding to the ith satellite and the jth baseline, n_{ij} is the integer ambiguity, λ is the wavelength of the GPS L_1 carrier signal, $\hat{\boldsymbol{s}}_i$ is the unit LOS vector of ith satellite, $\boldsymbol{b}_j$ is the jth baseline vector, β_j is the line bias corresponding to the jth baseline, and w_{ij}^2 is the weighted factor.

The typical method of attitude determination can be divided into two independent steps: resolving the integer ambiguities and determining attitude matrix $\boldsymbol{A}$. Once the integer ambiguities are fixed, they no longer need to be resolved in the later procedure. Therefore,

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the following equivalent differential range can be defined if the integer ambiguities are known:

$$\Delta r_{ij} = \Delta\phi_{ij} + n_{ij} \cdot \lambda - \beta_j \tag{2}$$

Substituting Eq. (2) into Eq. (1), we obtain

$$J(\boldsymbol{A}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \left(\Delta r_{ij} - \hat{\boldsymbol{s}}_i^T \boldsymbol{A}^T \boldsymbol{b}_j\right)^2 \tag{3}$$

One kind of candidate to resolve the integer ambiguities is the motion-based method,[7–9] including both platform motion-based methods and GPS satellite motion-based methods. Both of them collect carrier phase measurements over a few epochs until the antenna platform moves or, obviously, the geometry of visible GPS satellites changes. Although the motion-based methods have been proven to be efficient, another kind of method, called the instantaneous method, is more suitable for real-time applications, that is, spacecraft attitude control. An effective instantaneous ambiguity resolution is the Knight method, which has been applied on the TANS Vector GPS receiver.[10] The Knight method has the capability of determining the ambiguities utilizing only the measurements at an epoch. Mathematically, it is based on the search principle, and it employs a Kalman filter to do the search procedure to find the most probable solution from a number of candidates.

Although the integer ambiguities solution is very crucial to the problem of attitude determination using GPS, the following discussion will focus on the method to resolve the attitude matrix and assume that the integer ambiguities have been resolved by a method. In fact, we employ the Knight method to resolve the integer ambiguities in the experiments.

The formulation in this paper is given in terms of single difference carrier phase measurement in which the line bias is constant. This would be the case for a receiver such as the TANS Vector. Most receivers have some kind of time-varying clock bias that must be determined along with the attitude. The antenna configuration is assumed rigid in this paper, and flexibility of the configuration is out of the scope of the present paper.

Traditional Approach

Two traditional approaches to the problem of Eq. (3) are the NLLSFit and the PWP. These are summarized in the following paragraphs.

NLLSFit Method

As is well known, the attitude matrix $\boldsymbol{A}$ can be expressed by a first-order linearization about a nominal attitude $\boldsymbol{A}_0$ as follows:

$$\boldsymbol{A} = [\boldsymbol{I} + \Omega(\delta\boldsymbol{\theta})]\boldsymbol{A}_0 \tag{4}$$

$$\Omega(\delta\boldsymbol{\theta}) = \begin{bmatrix} 0 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 0 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 0 \end{bmatrix} \tag{5}$$

where $\delta\theta_i$, $i = 1, 2, 3$, are the three components of $\delta\boldsymbol{\theta}$.

To define $\Omega(\boldsymbol{b}_j)$, as in Eq. (5), only using $\boldsymbol{b}_j$ instead of $\delta\boldsymbol{\theta}$, Eq. (3) can be converted to the following form:

$$J(\boldsymbol{A}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \left[\Delta r_{ij} - \hat{\boldsymbol{s}}_i^T \boldsymbol{A}_0^T \boldsymbol{b}_j - \hat{\boldsymbol{s}}_i^T \boldsymbol{A}_0^T \Omega(\boldsymbol{b}_j)\delta\boldsymbol{\theta}\right]^2 \tag{6}$$

When the variables z_{ij} and a (1×3) matrix $\boldsymbol{h}_{ij}$ are introduced as

$$z_{ij} = \Delta r_{ij} - \hat{\boldsymbol{s}}_i^T \boldsymbol{A}_0^T \boldsymbol{b}_j \tag{7}$$

$$\boldsymbol{h}_{ij} = \hat{\boldsymbol{s}}_i^T \boldsymbol{A}_0^T \Omega(\boldsymbol{b}_j) \tag{8}$$

then the linear weighted least-squares solution of Eq. (6) is given by

$$\delta\hat{\boldsymbol{\theta}} = \left[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \boldsymbol{h}_{ij}^T \boldsymbol{h}_{ij}\right]^{-1} \left[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \boldsymbol{h}_{ij}^T z_{ij}\right] \tag{9}$$

From Eq. (4), the nominal $\boldsymbol{A}_0$ can be corrected by $\delta\hat{\boldsymbol{\theta}}$:

$$\hat{\boldsymbol{A}} = [\boldsymbol{I} + \Omega(\delta\hat{\boldsymbol{\theta}})]\boldsymbol{A}_0$$

The new estimation of $\boldsymbol{A}$ is used to compute a new correction angle for the next iteration until $\|\delta\hat{\boldsymbol{\theta}}\|$ is smaller than a certain specified accuracy.

PWP

This approach originated from Cohen and Parkinson's work and is summarized as follows.[7] Using baseline vectors and LOS vectors, we define the matrices $\boldsymbol{B}(3 \times m)$ and $\boldsymbol{S}(3 \times n)$:

$$\boldsymbol{B} = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_m] \tag{10}$$

$$\boldsymbol{S} = [\hat{\boldsymbol{s}}_1 \quad \hat{\boldsymbol{s}}_2 \quad \cdots \quad \hat{\boldsymbol{s}}_n] \tag{11}$$

We also construct $\Delta\boldsymbol{R}(m \times n)$ of single difference range measurements:

$$\Delta\boldsymbol{R} = \begin{bmatrix} \Delta r_{11} & \Delta r_{12} & \cdots & \Delta r_{1n} \\ \Delta r_{21} & \Delta r_{22} & \cdots & \Delta r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta r_{m1} & \Delta r_{m2} & \cdots & \Delta r_{mn} \end{bmatrix} \tag{12}$$

If at the same time we introduce two positive numbers w_{sii}^2, which is the (i, i)th element of a diagonal positive definite matrix $\boldsymbol{W}_s(n \times n)$, and w_{bjj}^2, which is the (j, j)th element of a diagonal positive definite matrix $\boldsymbol{W}_B(m \times m)$, such that

$$w_{ij}^2 = w_{sii}^2 \cdot w_{bjj}^2 \tag{13}$$

which replaces the weighted factor w_{ij}^2, where w_{sii}^2 and w_{bjj}^2 can be regarded as the weighted factors associated with LOS vectors and baselines, respectively, then the problem of Eq. (3) can be converted into the following equivalent form[7]:

$$J(\boldsymbol{A}) = \left\|\boldsymbol{W}_B^{\frac{1}{2}}(\Delta\boldsymbol{R} - \boldsymbol{B}^T\boldsymbol{A}\boldsymbol{S})\boldsymbol{W}_S^{\frac{1}{2}}\right\|_F^2 \tag{14}$$

where $\|\cdot\|_F^2 = \mathrm{tr}([\cdot]^T[\cdot])$ is the Frobenius norm of a matrix, $\boldsymbol{W}_B^{1/2}$ is the square root of $\boldsymbol{W}_B$, and $\boldsymbol{W}_S^{1/2}$ is the square root of $\boldsymbol{W}_S$. If $\boldsymbol{W}_B$ is chosen so as to satisfy the condition

$$\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T = \boldsymbol{I} \tag{15}$$

then the form of Eq. (14) can be made identical to that of Wahba's problem, which maximizes the new cost function:

$$J'(\boldsymbol{A}) = \mathrm{tr}\left(\boldsymbol{A}\boldsymbol{S}\boldsymbol{W}_S\Delta\boldsymbol{R}^T\boldsymbol{W}_B\boldsymbol{B}^T\right) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{G}^T) \tag{16}$$

where $\boldsymbol{G} = \boldsymbol{B}\boldsymbol{W}_B\Delta\boldsymbol{R}\boldsymbol{W}_S\boldsymbol{S}^T$. Equation (15) is referred to as Cohen and Parkinson's balanced condition.[7] There are several methods to solve Eq. (16), such as QUEST,[11] the Euler-q method (see Ref. 12), and so on. In fact, by introducing quaternion $\tilde{\boldsymbol{q}}$ to express the attitude matrix, the problem of Eq. (16) can be identical to one of finding the optimal solution $\tilde{\boldsymbol{q}}_{\mathrm{opt}}$, which is the eigenvector of a (4×4) matrix $\boldsymbol{K}$ corresponding to the maximum eigenvalue of $\boldsymbol{K}$, $\mu_{\max}$. Thus,

$$\boldsymbol{K}\tilde{\boldsymbol{q}}_{\mathrm{opt}} = \mu_{\max}\tilde{\boldsymbol{q}}_{\mathrm{opt}} \tag{17}$$

where

$$\boldsymbol{K} = \left[\begin{array}{c:c} \boldsymbol{G} + \boldsymbol{G}^T - \mathrm{tr}(G)\cdot\boldsymbol{I}_{3\times 3} & \boldsymbol{v} \\ \hdashline \boldsymbol{v}^T & \mathrm{tr}(\boldsymbol{G}) \end{array}\right]$$

and $\boldsymbol{v}$ is a (3×1) vector defined as

$$v_1 = \boldsymbol{G}_{23} - \boldsymbol{G}_{32}, \qquad v_2 = \boldsymbol{G}_{31} - \boldsymbol{G}_{13}, \qquad v_3 = \boldsymbol{G}_{12} - \boldsymbol{G}_{21}$$

where $\boldsymbol{v}_j$ is the jth element of $\boldsymbol{v}$, and $\boldsymbol{G}_{ij}$ is the (i, j)th element of $\boldsymbol{G}$.

Up to this point, the optimal solution $\tilde{\boldsymbol{q}}_{\mathrm{opt}}$ has been obtained, and the attitude matrix $\boldsymbol{A}$ may also be calculated by $\tilde{\boldsymbol{q}}_{\mathrm{opt}}$. However, the condition of Eq. (15) is referred to as a balanced baseline configuration. It is evident that the balanced condition is the necessary condition for Eq. (3) to be identical to Wahba's problem, Eq. (16). Obviously, at least three noncoplanar baselines are necessary to satisfy the balanced condition of Eq. (15). For the case of an unbalanced configuration, some improved methods have been presented in the literature.[4–6]

New Approach

Although the problem of Eq. (3) is a nonlinear function of the attitude angles, it is a linear function of the attitude matrix $\boldsymbol{A}$. The key idea is to convert the problem of Eq. (3) to a weighted least-squares problem of elements of $\boldsymbol{A}$. Introduce a (9×1) state vector $\boldsymbol{a}$ to express $\boldsymbol{A}$ as follows:

$$\boldsymbol{a} = \left[\boldsymbol{a}_1^T \;\vdots\; \boldsymbol{a}_2^T \;\vdots\; \boldsymbol{a}_3^T\right]^T \tag{18}$$

where $\boldsymbol{a}_i^T$, $i = 1, 2, 3$, is the ith row of $\boldsymbol{A}$. Then

$$\hat{\boldsymbol{s}}_i^T \boldsymbol{A}^T \boldsymbol{b}_j = \left[b_{jx}\hat{\boldsymbol{s}}_i^T \;\vdots\; b_{jy}\hat{\boldsymbol{s}}_i^T \;\vdots\; b_{jz}\hat{\boldsymbol{s}}_i^T\right] \begin{bmatrix} \boldsymbol{a}_1 \\ \cdots \\ \boldsymbol{a}_2 \\ \cdots \\ \boldsymbol{a}_3 \end{bmatrix} = \tilde{\boldsymbol{h}}_{ij}\boldsymbol{a} \tag{19}$$

where b_{jx}, b_{jy}, and b_{jz} are the three components of vector $\boldsymbol{b}_j$, and

$$\tilde{\boldsymbol{h}}_{ij} = \left[b_{jx}\hat{\boldsymbol{s}}_i^T \;\vdots\; b_{jy}\hat{\boldsymbol{s}}_i^T \;\vdots\; b_{jz}\hat{\boldsymbol{s}}_i^T\right]$$

is a (1×9) matrix.

Substituting Eq. (19) into Eq. (3), we obtain

$$J(\boldsymbol{a}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2(\Delta r_{ij} - \tilde{\boldsymbol{h}}_{ij}\boldsymbol{a})^2 \tag{20}$$

When the matrix

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \tilde{\boldsymbol{h}}_{ij}^T \tilde{\boldsymbol{h}}_{ij}\right]$$

is nonsingular, the weighted least-squares solution of Eq. (20) is given by

$$\hat{\boldsymbol{a}} = \left[\sum_{j=1}^{m} w_{bjj}^2 \sum_{i=1}^{n} w_{sii}^2 \tilde{\boldsymbol{h}}_{ij}^T \tilde{\boldsymbol{h}}_{ij}\right]^{-1} \left[\sum_{j=1}^{m} w_{bjj}^2 \sum_{i=1}^{n} w_{sii}^2 \tilde{\boldsymbol{h}}_{ij}^T \Delta r_{ij}\right] \tag{21}$$

Taking into account Eq. (19), we obtain

$$\tilde{\boldsymbol{h}}_{ij}^T \tilde{\boldsymbol{h}}_{ij} = \begin{bmatrix} b_{jx}^2\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jx}b_{jy}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jx}b_{jz}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) \\ b_{jy}b_{jx}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jy}^2\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jy}b_{jz}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) \\ b_{jz}b_{jx}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jz}b_{jy}\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) & b_{jz}^2\left(\hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\right) \end{bmatrix} \tag{22}$$

By applying Eq. (13) and introducing the following vectors,

$$\hat{\boldsymbol{u}}_j = \left(\sum_{i=1}^{n} w_{sii}^2 \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T\right)^{-1} \left(\sum_{i=1}^{n} w_{sii}^2 \hat{\boldsymbol{s}}_i \Delta r_{ij}\right) \tag{23}$$

where $j = 1, 2, \ldots, m$, Eq. (21) can be proven to split into three parts so that the rows of $\boldsymbol{A}$ can be estimated separately as follows:

$$\hat{\boldsymbol{a}}_i = \sum_{j=1}^{m} w_{bjj}^2 \left(\boldsymbol{d}_i^T \boldsymbol{b}_j\right)\hat{\boldsymbol{u}}_j, \qquad i = 1, 2, 3 \tag{24}$$

where vector $\boldsymbol{d}_i$ is the ith row of the following 3×3 matrix,

$$\boldsymbol{D} = \left(\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T\right)^{-1} \tag{25}$$

As in the preceding section, $\boldsymbol{W}_B$ is a $(m \times m)$ diagonal positive matrix of which the (j, j)th element is w_{bjj}^2. It can be proven that $(\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T)$ is nonsingular when the matrix

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2 \tilde{\boldsymbol{h}}_{ij}^T \tilde{\boldsymbol{h}}_{ij}\right]$$

is nonsingular, and vise versa. The case that $(\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T)$ is singular usually implies a coplanar configuration, which will be discussed in the succeeding section.

If a baseline configuration just exactly satisfies the following condition, which, hereafter, will be named the symmetric condition,

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx}b_{jy} = 0, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jx}b_{jz} = 0$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jy}b_{jz} = 0 \tag{26}$$

then Eq. (24) can be further simplified as

$$\hat{\boldsymbol{a}}_1 = \frac{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jx}\hat{\boldsymbol{u}}_j\right)}{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jx}^2\right)} \tag{27a}$$

$$\hat{\boldsymbol{a}}_2 = \frac{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jy}\hat{\boldsymbol{u}}_j\right)}{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jy}^2\right)} \tag{27b}$$

$$\hat{\boldsymbol{a}}_3 = \frac{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jz}\hat{\boldsymbol{u}}_j\right)}{\sum_{j=1}^{m}\left(w_{bjj}^2 b_{jz}^2\right)} \tag{27c}$$

Thus far, the attitude matrix $\boldsymbol{A}$ can be calculated using Eq. (24). Moreover, if the condition of Eq. (26) is satisfied, $\boldsymbol{A}$ can be simplified to be computed from Eqs. (27a–27c). Note that, because the orthogonal constraint of $\boldsymbol{A}$ has not been taken into account, the solution of either Eq. (24) or Eqs. (27a–27c) is not necessarily orthogonal.

Symmetric Condition

The configuration that satisfies the symmetric condition of Eq. (26) is here termed the symmetric baseline configuration. Figure 1 shows some examples of symmetric baseline configuration, where M represents the master antenna and i represents the ith slave antenna. One can use the following method to determine if a configuration is symmetric or not:

If there exists a positive $(m \times m)$ diagonal marix $\boldsymbol{W}_B$ such that

$$\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T = \Lambda \tag{28}$$

where Λ is an arbitrary (3×3) nonnegative diagonal matrix, m is the number of baselines, and $\boldsymbol{B}$ is the baseline matrix as shown in Eq. (10), then the configuration is a symmetric baseline configuration.

This can be easily proved, as follows. Expanding the left-hand side of Eq. (28), we obtain

$$\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T = \sum_{j=1}^{m} w_{bjj}^2 \begin{bmatrix} b_{jx}^2 & b_{jx}b_{jy} & b_{jx}b_{jz} \\ b_{jy}b_{jx} & b_{jy}^2 & b_{jy}b_{jz} \\ b_{jz}b_{jx} & b_{jz}b_{jy} & b_{jz}^2 \end{bmatrix} \tag{29}$$

Substituting Eq. (29) into Eq. (28), we derive the following equations:

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx}^2 = \Lambda_{11}, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jy}^2 = \Lambda_{22}$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jz}^2 = \Lambda_{33} \tag{30a}$$

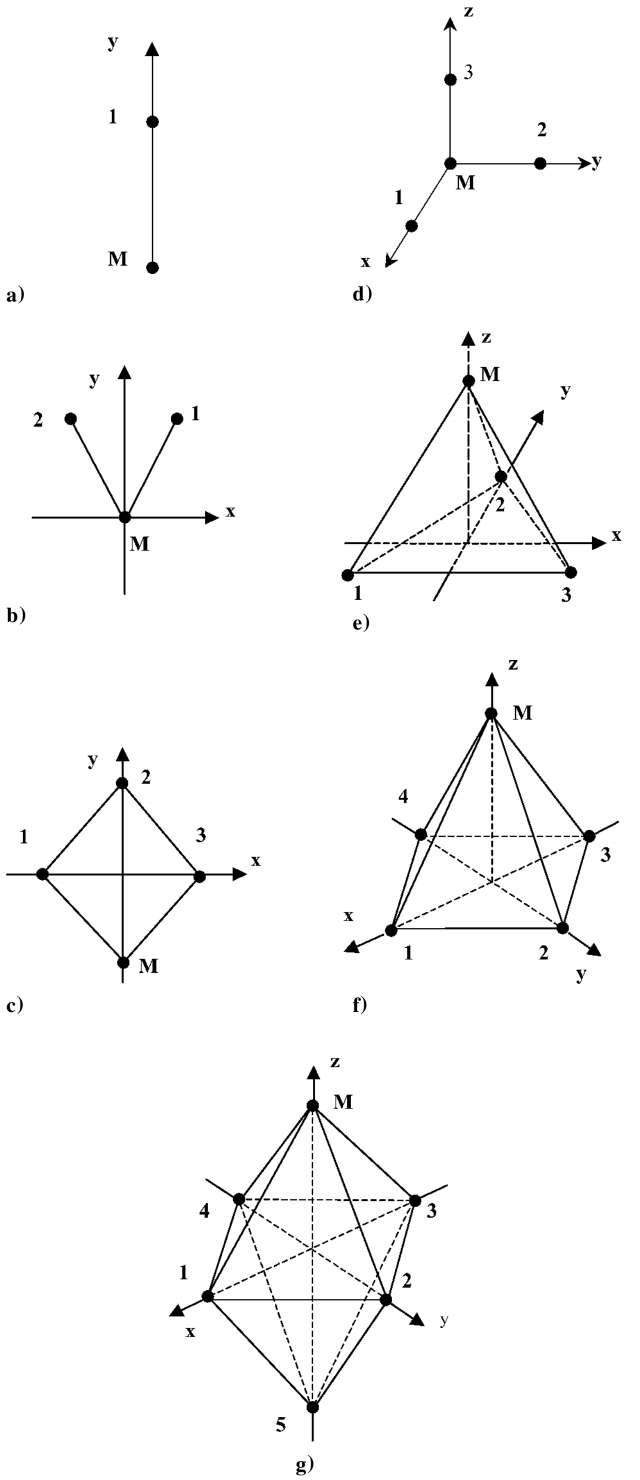

Fig. 1 Examples of symmetric baseline configuration.

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jy} = 0, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jz} = 0$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jy} b_{jz} = 0 \tag{30b}$$

Because the left-hand side of Eq. (30a) is always nonnegative and Λ is an arbitrary (3×3) nonnegative diagonal matrix, Eq. (30a) is always right. Clearly, Eq. (28) is identical to Eq. (30b).

If the matrix Λ happens to be a (3×3) unit matrix, Eq. (28) will be the same as Eq. (15). Therefore, the symmetric condition can be regarded as the generalization of the balanced condition. Substituting Eq. (29) into Eq. (15), we derive the following equations to show that the symmetric condition differs from the balanced condition:

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx}^2 = 1, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jy}^2 = 1, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jz}^2 = 1 \tag{31a}$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jy} = 0, \qquad \sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jz} = 0$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jy} b_{jz} = 0 \tag{31b}$$

It is evident that Eq. (31b) is the same as Eq. (30b). As pointed out, the symmetric condition only requires that we satisfy Eq. (30b). However, the balanced condition requires that we satisfy both Eqs. (31a) and (31b) simultaneously.

From Eq. (31a), it is evident that b_{jz}, $j = 1, 2, \ldots, m$, cannot be zero simultaneously. It implies that a coplanar baseline configuration can never satisfy the balanced condition. However, there is no such restriction for the symmetric condition. Because of this, the concept of a symmetric condition is applicable for any kind of configuration, including both coplanar and noncoplanar configurations and two-antenna configurations. As shown earlier and pointed out again here, the symmetric configuration is not the necessary condition of the AMES solution and simply acts to simplify the derived solution. However, the balanced condition is the necessary condition for the PWP method to convert equivalently the GPS attitude determination problem into Wahba's problem.

New Algorithm for Coplanar Baseline Configurations

The symmetric condition is more easily satisfied by coplanar baseline configurations than by noncoplanar baseline configurations. Because the z components of all baseline vectors $\boldsymbol{b}_j$ are zero, the last two equations of Eq. (26) are always satisfied. For the same reason, the matrix $(\boldsymbol{B}\boldsymbol{W}_B\boldsymbol{B}^T)$ becomes singular in a coplanar configuration situation. This means that the nine-dimensional state $\boldsymbol{a}$ cannot be estimated by Eq. (24). Fortunately, in this case, the problem of Eq. (20) can be reduced to a six-parameter criterion, which minimizes the following cost function:

$$J(\boldsymbol{a}_1, \boldsymbol{a}_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}^2 \left(\Delta r_{ij} - b_{jx} \hat{\boldsymbol{s}}_i^T \boldsymbol{a}_1 - b_{jy} \hat{\boldsymbol{s}}_i^T \boldsymbol{a}_2 \right)^2 \tag{32}$$

Because $\boldsymbol{A}$ is an orthogonal matrix, $\boldsymbol{a}_3$ can be derived as follows, once $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ have been estimated:

$$\hat{\boldsymbol{a}}_3 = \hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2 \tag{33}$$

The weighted least-squares solution of Eq. (32) can be deduced as follows:

$$\hat{\boldsymbol{a}}_i = \sum_{j=1}^{m} w_{bjj}^2 (e_{i1} b_{jx} + e_{i2} b_{jy}) \hat{\boldsymbol{u}}_j, \qquad i = 1, 2 \tag{34}$$

where $\hat{\boldsymbol{u}}_j$ is shown in Eq. (23) and

$$e_{11} = \frac{1}{e} \sum_{j=1}^{m} w_{bjj}^2 b_{jy}^2 \tag{35a}$$

$$e_{12} = e_{21} = -\frac{1}{e} \sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jy} \tag{35b}$$

$$e_{22} = \frac{1}{e} \sum_{j=1}^{m} w_{bjj}^2 b_{jx}^2 \tag{35c}$$

$$e = \sum_{j=1}^{m} w_{bjj}^2 b_{jx}^2 \cdot \sum_{j=1}^{m} w_{bjj}^2 b_{jy}^2 - \left(\sum_{j=1}^{m} w_{bjj}^2 b_{jx} b_{jy} \right)^2 \tag{35d}$$

Under the symmetric condition, they can be simplified in the same way Eqs. (27a) and (27b). The AMES algorithm for two typical symmetric configurations is formulated as follows.

Three-Antenna Configuration

The typical symmetric baseline configuration of three antennae is shown in Fig. 1b. If the lengths of two baselines are the same, their positions in the antenna coordinate system are given as

$$\boldsymbol{b}_1 = [b_x \quad b_y \quad 0]^T, \qquad \boldsymbol{b}_2 = [-b_x \quad b_y \quad 0]^T$$

It is easy to show that the configuration is symmetric by simply setting $w_{b11} = w_{b22} = 1$. According to Eqs. (27a) and (27b), the AMES solution in this case is then

$$\hat{\boldsymbol{a}}_1 = (\hat{\boldsymbol{u}}_1 - \hat{\boldsymbol{u}}_2)/2b_x \tag{36a}$$

$$\hat{\boldsymbol{a}}_2 = (\hat{\boldsymbol{u}}_1 + \hat{\boldsymbol{u}}_2)/2b_y, \qquad \hat{\boldsymbol{a}}_3 = \hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2 \tag{36b}$$

Four-Antenna Configuration

The typical symmetric baseline configuration of four antennae arranged in a square is shown in Fig. 1c. The positions of three baselines in the antenna coordinate system are

$$\boldsymbol{b}_1 = [-b \quad b \quad 0]^T, \qquad \boldsymbol{b}_2 = [0 \quad 2b \quad 0]^T$$

$$\boldsymbol{b}_3 = [b \quad b \quad 0]^T$$

It is evident that the configuration is symmetric by setting $w_{b11} = w_{b22} = w_{b33} = 1$. According to Eqs. (27a) and (27b), the AMES solution in this case is then

$$\hat{\boldsymbol{a}}_1 = (\hat{\boldsymbol{u}}_3 - \hat{\boldsymbol{u}}_1)/2b \tag{37a}$$

$$\hat{\boldsymbol{a}}_2 = (\hat{\boldsymbol{u}}_1 + 2\hat{\boldsymbol{u}}_2 + \hat{\boldsymbol{u}}_3)/6b, \qquad \hat{\boldsymbol{a}}_3 = \hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2 \tag{37b}$$

Compass Algorithm

The two-antenna configuration always satisfies the symmetric condition, as shown in Fig. 1a. Although only a row of $\boldsymbol{A}$ can be estimated, the baseline vectors separated by two antennas act like a compass and are able to determine two attitude angles. In this case, the problem of Eq. (20) is further reduced to a three-parameter criterion, which minimizes the following cost function:

$$J(\boldsymbol{a}_2) = \sum_{i=1}^{n} w_{sii}^2 \left(\Delta r_i - b\hat{\boldsymbol{s}}_i^T \boldsymbol{a}_2\right)^2 \tag{38}$$

where b is the length of the baseline and Δr_i is the equivalent differential range associated with ith satellite.

The weighted least-squares solution of the preceding problem is as follows:

$$\hat{\boldsymbol{a}}_2 = \frac{1}{b}\left[\sum_{i=1}^{n} w_{sii}^2 \left(\hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T\right)\right]^{-1} \left[\sum_{i=1}^{n} w_{sii}^2 \hat{\boldsymbol{s}}_i \Delta r_i\right] \tag{39}$$

Suppose $\boldsymbol{A}$ is the 3(ψ)-1(θ)-2(γ) sequence as follows:

$$\boldsymbol{A} = \begin{bmatrix} c\gamma \cdot c\psi - s\gamma \cdot s\theta \cdot s\psi & c\gamma \cdot s\psi + s\gamma \cdot s\theta \cdot c\psi & -s\gamma \cdot c\theta \\ -c\theta \cdot s\psi & c\theta \cdot c\psi & s\theta \\ s\gamma \cdot c\psi + c\gamma \cdot s\theta \cdot s\psi & s\gamma \cdot s\psi - c\gamma \cdot s\theta \cdot c\psi & c\gamma \cdot c\theta \end{bmatrix} \tag{40}$$

where ψ is azimuth, θ is pitch, γ is roll, c is the cosine function, and s is the sine function.

The azimuth and pitch are given by

$$\psi = -\tan^{-1}(a_{21}/a_{22}), \qquad \theta = \tan^{-1}\left(a_{23}\Big/\sqrt{a_{21}^2 + a_{22}^2}\right) \tag{41}$$

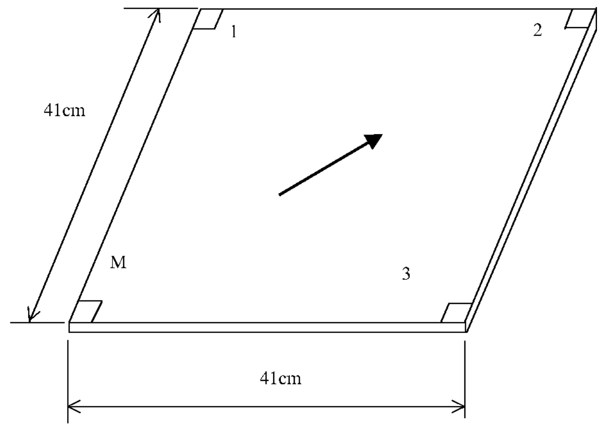

Fig. 2 TANS Vector's four-antenna square configuration.

Experiments

This section presents some results of experiments performed to test the new algorithm. The raw single difference carrier phase measurements and LOS vectors were received from a TANS Vector GPS receiver, which is a solid-state attitude determination and position location system with a four-antenna array. Figure 2 shows four antennas arranged in a 41 × 41 cm square. The definition of the antenna coordinate system is also shown in Fig. 1c. It is easy to understand how this configuration satisfies the symmetric condition. The positions of three baselines in the antenna coordinate system are

$$\boldsymbol{b}_1 = [-b \quad b \quad 0]^T, \qquad \boldsymbol{b}_2 = [0 \quad 2b \quad 0]^T$$

$$\boldsymbol{b}_3 = [b \quad b \quad 0]^T$$

where $b = 29$ cm.

One of experiments was carried out on 23 December 1998. The experiment was conducted continuously for about 1 h, from GPS time 259360.75 to 262780.75 s. The LOS vectors and single difference carrier phase measurements were recorded so that they could be utilized in postprocessing. The single difference carrier phase measurements were output at about 2.0 s intervals. As mentioned, the integer ambiguities were resolved by the Knight method.[10]

There are always more than four satellites tracked during the experiment. During about 87% of the period, there are six satellites tracked. Because at least four visible satellites ($n \geq 4$) are necessary for navigation, algorithms are permitted to work when there are more than four satellites being processed. In fact, it is easy to see from Eq. (23) that AMES requires at least three visible satellites ($n \geq 3$). However, NLLSFit only requires $n \geq 2$.

Coplanar Baseline Configuration

Three cases are considered: 1) coplanar three-baseline configuration ($\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$), 2) two-baseline configuration ($\boldsymbol{b}_1$ and $\boldsymbol{b}_3$), and 3) one-baseline configuration ($\boldsymbol{b}_2$). AMES and NLLSFit are employed in the first case, and in the second case they are joined by the TRIAD[13] algorithm. Among these three algorithms, only AMES is available in the third case to determine two angles: azimuth and pitch. An IBM ThinkPad portable computer with an Intel Pentium III CPU processes the recorded data. The absolute cost of an algorithm is dependent on many factors, such as the coding, CPU, and the accuracy constraint if it is an iterative procedure. Here, only NLLSFit needs an iterative procedure. In this investigation, the iterative accuracy is set to less than 10^{-3} rad with the maximum iteration of five. NLLSFit took only two iterations to reach the required processing accuracy. The results are listed in Table 1, which lists the standard deviation (STD) and the average computational time of each algorithm. The solution of AMES for three-baseline configuration is shown in Fig. 3, where the multipath is evidently serious.

From Table 1, it is evident that the difference of STDs between NLLSFit and AMES is very small. Both are smaller than that of the TRIAD algorithm. In the three-baseline case, the computational burden of AMES is about 38.4% of that of NLLSFit and about 40.5% in

Table 1 Results for coplanar configurations

Employed baselines	Algorithm	STD, deg			Computation time, ms
		Azimuth	Pitch	Roll	
(b_1, b_2, b_3)	AMES	0.415	0.305	0.185	0.038
	NLLSFit	0.417	0.291	0.172	0.099
(b_1, b_3)	TRIAD	0.699	0.447	0.185	0.023
	AMES	0.477	0.447	0.185	0.034
	NLLSFit	0.484	0.436	0.170	0.084
(b_2)	AMES	0.507	0.257	——	0.020

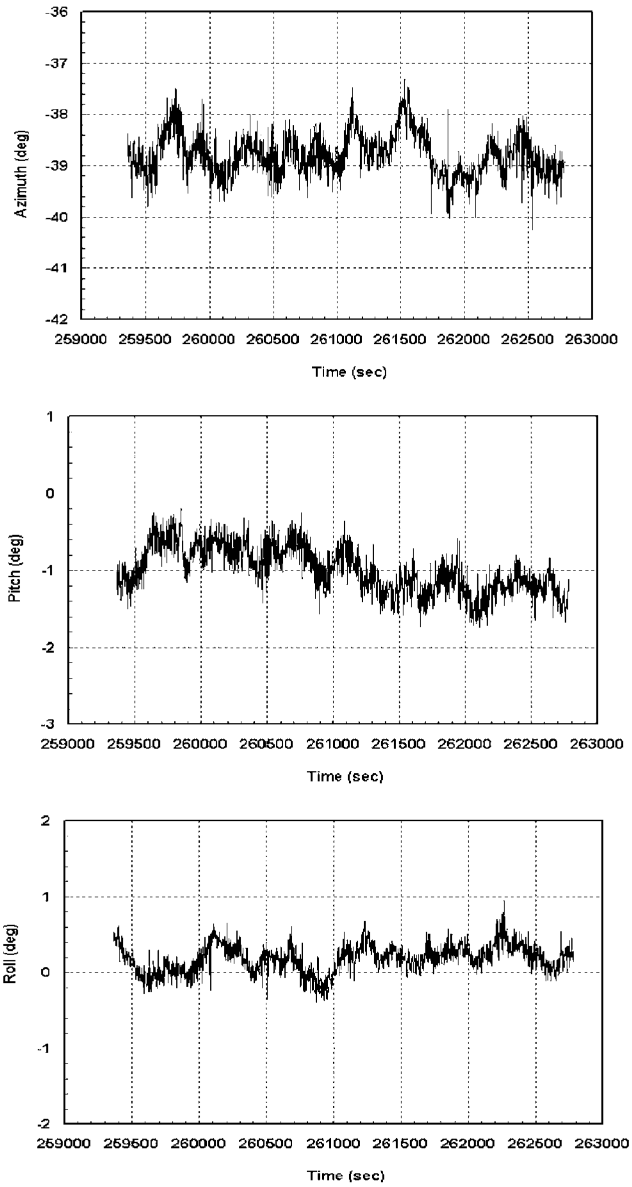

Fig. 3 Solution of AMES for coplanar three-baseline configuration.

the two-baseline case. The results presented in Table 1 are obtained from the solution of Eq. (34), which is suited to arbitrary coplanar baseline configurations. As shown earlier, if taking into account the symmetric condition, the solution can be further simplified as was that of Eqs. (27a) and (27b). This means that the computational cost of AMES can be further reduced. The result shows that it is reduced to 0.029 ms for the three-baseline case and 0.028 ms for the two-baseline case, to save about a 23.7% and a 17.6% burden, respectively. After taking into account the symmetric condition, the computational cost of AMES is about 29.3% of that of NLLSFit in the three-baseline case.

Note that the cost of NLLSFit listed in Table 1 only reflects iterative procedure costs. NLLSFit needs an initial attitude solution that may be obtained from a coarse but fast algorithm to start up the iterative procedure. It is reasonable that the burden of this coarse algorithm is accumulated into the burden of NLLSFit. For example, the TRIAD algorithm is utilized to obtain an initial solution. It costs 0.023 ms (see Table 1). For the three-baseline case, the total cost of NLLSFit will accumulate to 0.122 ms, the sum of the cost of the TRIAD algorithm and the cost of the iterative procedure. In this case, the cost of AMES is only about 31.1% of that of NLLSfit.

Orthogonalization

As shown, the derived AMES solution, either Eq. (24) or Eq. (34), is the constraint-free solution in which the following orthogonal constraint is not taken into account[14]:

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I} \tag{42}$$

This implies that an additional orthogonalization procedure can improve the solution of AMES. For example, by considering $\hat{\boldsymbol{A}}$, the solution of AMES being close to an orthogonal matrix, one can adopt one of two orthogonalization procedures, as follows,[14]

$$\hat{\boldsymbol{A}}^{(c)} = (\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^{-T})/2 \tag{43}$$

or

$$\hat{\boldsymbol{A}}^{(c)} = (3\boldsymbol{I} + \hat{\boldsymbol{A}}\hat{\boldsymbol{A}}^T)\hat{\boldsymbol{A}}/2 \tag{44}$$

One also can employ a baseline length fixing procedure to improve the accuracy of solution. In fact, $\hat{\boldsymbol{u}}_j$ in Eq. (23) is the jth baseline vector in the reference coordinate system. It should be of the same length as $\boldsymbol{b}_j$. Therefore, $\hat{\boldsymbol{u}}_j$ can be improved such that

$$\hat{\boldsymbol{u}}_j^{(c)} = \frac{\hat{\boldsymbol{u}}_j \cdot |\boldsymbol{b}_j|}{|\hat{\boldsymbol{u}}_j|} \tag{45}$$

where $|\cdot|$ is the length of the vector. The corrected solution $\hat{\boldsymbol{u}}_j^{(c)}$ would be applied in the AMES solution to replace $\hat{\boldsymbol{u}}_j$.

The orthogonalization procedure can improve the accuracy of the azimuth solution. When Eq. (44) is employed, the accuracy of azimuth can be improved from 0.415 to 0.405 deg, about a 2.4% improvement. However, the computational time increases 0.0044 ms, leading to additional burden of about 11.6%. Thus, AMES is more efficient and practical when executing without an orthogonalization process. Note that the results in Table 1 are those without applying the orthogonalization procedure.

Noncoplanar Baseline Configuration

AMES is compared with PWP and NLLSFit in this section. A simulation method called semimathematical simulation is used to generate single-difference carrier phase measurements for a noncoplanar baseline configuration. The measurement errors and LOS vectors are completely the same as the actual data taken from an experiment with the coplanar baseline configuration (see Fig. 2). Because the baseline length is short enough, errors in the measurements can be assumed to be almost equal when the baseline configuration is changed. With this assumption, errors can be simulated by

$$\varepsilon_{ij} = \Delta\phi_{ij} + n_{ij} \cdot \lambda - \hat{\boldsymbol{s}}_i^T \boldsymbol{A}^T \boldsymbol{b}_j \tag{46}$$

Thus, the measurement data are computed by the sum of a new baseline's projection on an LOS vector and the measurement errors as

$$\Delta\phi_{ij}^{\text{new}} = \hat{\boldsymbol{s}}_i^T \boldsymbol{A}^T \boldsymbol{b}_j^{\text{new}} - n_{ij}^{\text{new}} \cdot \lambda + \varepsilon_{ij} \tag{47}$$

The noncoplanar baseline configuration employed in the simulation is defined as follows:

$$\boldsymbol{b}_1 = [-b \quad b \quad 0]^T, \qquad \boldsymbol{b}_2 = [0 \quad 2b \quad h]^T$$

$$\boldsymbol{b}_3 = [b \quad b \quad 0]^T$$

where $b = \sqrt{2}/2$ m and $h = 0.5$ m.

Table 2 Results for a noncoplanar configuration

Algorithm	STD, deg			Computation time, ms
	Azimuth	Pitch	Roll	
AMES	0.196	0.184	0.076	0.043
NLLSFit	0.172	0.117	0.079	0.098
PWP	0.210	0.186	0.178	0.058

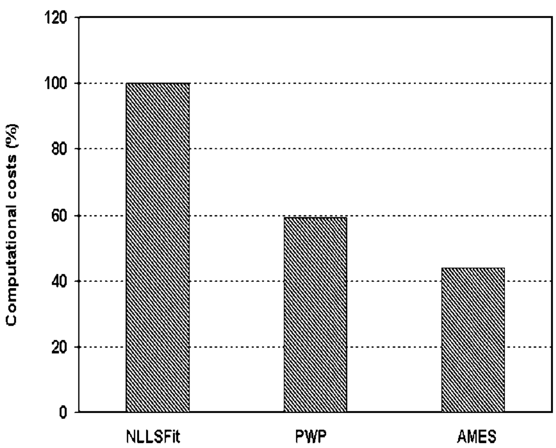

Fig. 4 Comparison of computation loads of algorithms.

Results of a simulation are listed in Table 2. As in Table 1, the cost of NLLSFit in Table 2 does not take into account the cost of the initial procedure, which obtains a coarse solution as a starting point of the iterative procedure. The burden of NLLSFit is about 56% heavier than that of AMES and about 40.8% heavier than that of PWP. These results are also depicted in Fig. 4, where the cost of NLLSFit acts as the reference. The total burden of NLLSFit will surely become heavier when accounting for the cost of an initial procedure.

For the case of unbalanced baselines, one can treat Eq. (14) as generalizing Wahba's problem where $\boldsymbol{W}_B$ and $\boldsymbol{W}_S$ are generalized as symmetric rather than diagonal symmetric matrices as suggested in Ref. 4. Thus, $\boldsymbol{W}_B = \boldsymbol{V}_B \boldsymbol{\Sigma}_B^{-2} \boldsymbol{V}_B^T$, where $\boldsymbol{\Sigma}_B$ and $\boldsymbol{V}_B$ come from the singular value decomposition (SVD) of $\boldsymbol{B}$ matrix such as $\boldsymbol{B} = \boldsymbol{U}_B \boldsymbol{\Sigma}_B \boldsymbol{V}_B^T$. However, especially for the case of three noncoplanar baselines, the matrix $\boldsymbol{B}$ in Eq. (10) is a 3×3 nonsingular matrix. In this case, one can use an inversion method to specify $\boldsymbol{W}_B$ such as $\boldsymbol{W}_B = (\boldsymbol{B}^T \boldsymbol{B})^{-1}$ to satisfy Eq. (15). The computational cost of this method is less than that of the SVD method, whereas the STDs of the two methods are almost the same. PWP results in Table 2 are obtained from the inversion method.

PWP converts identically the problem of Eq. (3) to that of Wahba's problem, the solution of which is optimum under the balanced configuration. It is evident that the configuration given is unbalanced. Thus, PWP gave a suboptimal solution. As mentioned, the AMES solution is also suboptimal because the orthogonal constraint is not taken into account. NLLSFit employs an iterative procedure to tend to the optimal solution. Practically, the iteration would be terminated when the accuracy converges to a required level. Therefore, the NLLSFit solution is the closest to optimum among the three algorithms. Thus, NLLSFit gives the best solution. This is demonstrated in Table 2. On the other hand, all of three algorithms employ different procedures to obtain their own results. Their accuracies are naturally different from each other.

According to the results given, AMES is more than twice as fast as NLLSFit, even if the burden of initial procedure has not been taken into account in the burden of NLLSFit. Although the performance of computer hardware has been improving greatly every month, the speed and accuracy are still crucial to an algorithm. Speed implies capability for a higher sampling rate, and it also means that an algorithm can save the resources of system. It is especially crucial in the situation where system resources are very limited, that is, the onboard computer of a control system of a spacecraft.

Conclusions

When the attitude matrix is converted into a state vector, a new approach is presented that efficiently resolves the problem of attitude determination using GPS. This new algorithm has a number of advantages that distinguish it considerably from traditional methods. First, the algorithm is just as easily realized in a computer as is creating a program for a standard weighted least squares method or a standard least squares method. Second, it does not require any initial attitude knowledge and avoids computationally expensive iteration. Last, it can be applied not only to both the coplanar and the noncoplanar baseline configurations, but also to the one-baseline configurations. Because of the symmetric baseline configuration, the solution of AMES can be further simplified, and the computational burden can be reduced correspondingly.

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